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A class of relativistic stars with a linear equation of state

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ABSTRACT
By assuming a particular mass function we find new exact solutions to the Einstein field equations with an anisotropic matter distribution. The solutions are shown to be relevant for the description of compact stars. A distinguishing feature of this class of solutions is that they admit a linear equation of state which can be applied to strange stars with quark matter.

Key words: Equation of state.

1 INTRODUCTION

In studies of compact relativistic astrophysical objects, strange stars have long been proposed as an alternative to neutron star models. The theoretical basis of strange stars composed of deconfined u, d and s quarks is a direct consequence of the conjecture that quark matter might be the true ground state of hadrons (Witten 1984; Farhi & Jaffe 1984). Strange stars are expected to form during the collapse of the core of a massive star after the supernova explosion. Another possibility is that a rapidly spinning neutron star can accrete sufficient mass to undergo a phase transition to become a strange star. Although we have no direct experimental evidence which may support the existence of such objects, recent observational data on the compactness (mass-radius ratio) of some of the compact objects like Her X-1, SAX J1808.4-3658, RX J185635-3754, 4U 1820-30, 4U 1728-34, PSR 0943+10, strongly favour the possibility that they could actually be strange stars (Bombaci 1997; Li *et al* 1995, 1999a,b; Dey *et al* 1998; Xu *et al* 1999, 2001; Pons *et al* 2002). Naturally, the internal composition and consequent geometry of such objects have become a subject of considerable interest over the last couple of decades. As the physics of very high density matter is still not very clear, most of the strange star studies have been performed within the framework of a bag model, see e.g., Glendenning *et al* (1995), Kettner *et al* (1995). In the phenomenological MIT bag model (Chodos *et al* 1974), it is assumed that the quark confinement is caused by a universal pressure B on the surface of any region containing quarks.

In the bag model, the strange matter equation of state (EOS) has a simple linear form given by $p = 1/3(\rho - 4B)$ (Witten 1984), where, ρ is the energy density, p is the isotropic pressure and B is the bag constant. Dey *et al* (1998) developed a new model for strange stars where the quark interaction is described by an interquark vector potential originating from gluon exchange and a density dependent scalar potential which restores chiral symmetry at a high density. The EOS formulated by Dey *et al* (1998) can also be approximated to a linear form $p = n(\rho - \rho_s)$, where, n and ρ_s are two parameters (Gondek-Rosińska *et al* 2000; Zdunik 2000). Thus, if pulsars are actually strange stars, linearity in EOS seems to be a feature in the composition of such objects. Once the EOS is known, for a given central density or pressure, the Tolman-Oppenheimer-Volkoff (TOV) equations are then integrated to calculate the macroscopic features such as mass and radius of the star.

However, as the densities within such stars stars are normally beyond nuclear matter density, one expects anisotropy to play a major role in such calculations as the TOV equations get modified in the presence of anisotropic pressures. Note that among many possibilities, one of the reasons for the consideration of anisotropy within such stars could be the presence of strong electric field as recently suggested by Usov (2004).

Since the initial paper by Bowers & Liang (1974) there has been extensive research in the study of anisotropic relativistic matter in general relativity. The analysis of static spherically symmetric anisotropic fluid spheres is important in relativistic astrophysics. The investigations of Ruderman (1972) indicated that nuclear matter may be anisotropic in high density ranges of order 10^{15} gm cm^{-3} where nuclear interactions have to be treated relativistically. Celestrial bodies are not composed of perfect fluids only: the radial pressure is not equal to the tangential pressure. Kippenhahn & Weigert (1990) showed that anisotropy could be introduced by the existence of a solid core or the presence of type 3A superfluid. Anisotropy can arise from different kinds of phase transitions (Sokolov 1980) or pion condensation (Sawyer 1972). Anisotropies in spherical galaxies, in the context of Newtonian gravity, have been studied by

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Binney & Tremaine (1987). The effects of slow rotation in a star (Herrera & Santos 1995) or the mixture of two gases (Letelier 1980) such as ionized hydrogen and electrons, can be modelled as a relativistic anisotropic fluid. Strong magnetic fields can generate an anisotropic pressure component inside a compact sphere as pointed out by Weber (1999).

Our analysis depends on two key assumptions. Firstly, we choose a rational functional form for the mass function. The form chosen ensures that the mass function is continuous and well-behaved in the interior of the star; it yields a monotonically decreasing energy density in the stellar interior. This is a desirable feature for the model on physical grounds. It is interesting to observe that many of the solutions found previously (Chaisi & Maharaj 2005, 2006) do not share this feature. It is important to note that the mass function chosen is similar to the profile that appears in the modelling of dark energy stars (Lobo 2006). Secondly, we choose an equation of state which linearly relates the radial pressure to the energy density. Maharaj & Chaisi (2006) have demonstrated that this simple equation of state is consistent in the modelling of compact matter such as neutron stars and quasi-stellar objects. The equation of state yields values of surface redshifts and masses that correspond to realistic stellar objects such as Her X-1 and Vela X-1 (Gokhroo & Mehra 1994; Chaisi & Maharaj 2005). Linear equations of state are relevant in the modelling of static spherically symmetric anisotropic quark matter distributions as demonstrated by Mak & Harko (2002) and anisotropic quark stars admitting a conformal symmetry (Mak & Harko 2004).

We attempt here to construct a new model for compact stars where pressure anisotropy is present and the EOS is linear. Our paper is organized as follows: In section 2, we develop a model for anisotropic stars with an in-built linear EOS. The physical analysis of the model will be discussed in section 3. We also perform some numerical calculations to show the role of anisotropy on the gross features of compact stars. In section 4, we summarize our results and point out avenues for further investigation.

2 ANISOTROPIC MODEL

We assume the line element for a static spherically symmetric anisotropic star in the standard form

$$ds^2 = -e^{\gamma(r)}dt^2 + e^{\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where $\gamma(r)$ and $\mu(r)$ are the two unknown metric functions. Assuming the energy momentum tensor for an anisotropic star in the most general form

$$T_{ij} = \mathrm{diag}\ (-\rho,\ p_r,\ p_t,\ p_t) \tag{2}$$

the field equations are obtained as

$$\rho = \frac{\left(1-e^{-\mu}\right)}{r^2} + \frac{\mu' e^{-\mu}}{r} \tag{3}$$

$$p_r = \frac{\gamma' e^{-\mu}}{r} - \frac{\left(1-e^{-\mu}\right)}{r^2} \tag{4}$$

$$p_t = \frac{e^{-\mu}}{4}\left(2\gamma'' + \gamma'^2 - \gamma'\mu' + \frac{2\gamma'}{r} - \frac{2\mu'}{r}\right) \tag{5}$$

where primes denote differentiation with respect to r. In equations (3)-(5), ρ is the energy density, p_r is the radial pressure and p_t is the tangential pressure. These equations may be combined to yield

$$(\rho + p_r)\gamma' + 2p_r' + \frac{4}{r}(p_r - p_t) = 0 \tag{6}$$

which is the conservation equation. If the mass contained within a radius r of the star is defined as

$$m(r) = \frac{1}{2}\int_0^r r'^2 \rho(r')dr' \tag{7}$$

we obtain the equivalent system of field equations

$$e^{-\mu} = 1 - \frac{2m(r)}{r} \tag{8}$$

$$\gamma' = \frac{2m(r) + p_r r^3}{r(r-2m)} \tag{9}$$

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{2m(r) + r^3 p_r}{2r(r-2m(r))} + \frac{2\Delta}{r} \tag{10}$$

where $\Delta = p_t - p_r$, is the measure of anisotropy in this model. Equation (10) is the modified TOV equation in the presence of anisotropy.

To make the above set of equations tractable, we choose the mass function in a particular form

$$m(r) = \frac{br^3}{2(1+ar^2)} \tag{11}$$

where a and b are two positive constants. The choice of the mass function is motivated by the fact that it gives a monotonically decreasing energy density in the stellar interior. Similar forms for the mass function have earlier been considered by Matese & Whitman (1980) and Finch & Skea (1989) for isotropic fluid spheres, Lobo (2006) for dark energy stars, and Mak & Harko (2003) for anisotropic fluid spheres.

Substituting equation (11) in equation (8), we obtain one of the metric functions in the form

$$e^{\mu} = \frac{1+ar^2}{1+(a-b)r^2}. \tag{12}$$

From equation (3), the energy density then can be obtained as

$$\rho = \frac{b(3+ar^2)}{(1+ar^2)^2} \tag{13}$$

so that the central density is given by $\rho_c = 3b$.

To find the other metric function $\gamma(r)$, in general, either a special functional form of the anisotropic parameter $\Delta(r)$ (see Mak & Harko (2003), Dev & Gleiser (2004)) or the pressure $p_r(r)$ profile (see Chaisi & Maharaj (2005)) is chosen at this point. However, since linearity seems to be a feature in compact star EOS (as shown by Sharma & Mukherjee (2001); Sharma *et al* (2002)), we assume a linear form of the EOS to solve equation (9). Integrating equation (9) we get

$$\gamma = \int \frac{p_r(1+ar^2)r}{1+(a-b)r^2}dr + \frac{b}{2(a-b)}\ln(1+ar^2-br^2) + \ln B \tag{14}$$

where, $\ln B$ is an integration constant. To evaluate the first term on right hand side of equation (14), we assume a linear EOS of the form

$$p_r = n(\rho - \rho_s), \tag{15}$$

where $0 \leqslant n \leqslant 1$ is a constant which is related to the sound speed $dp_r/d\rho = n$, and ρ_s is the density at the surface $r = s$, where the radial pressure vanishes. We do not impose any restrictions on the tangential pressure p_t. We obtain the other metric function in the form

$$e^{\gamma} = B(1+ar^2)^n(1+ar^2-br^2)^k \exp\left[-\frac{acr^2}{2(a-b)}\right] \tag{16}$$

where

$$k = \frac{5nab - 2na^2 - 3nb^2 + ab - b^2 - bc}{2(a-b)^2}. \tag{17}$$

In equation (17) we have set $c = n\rho_s$. Finally using equation (10), we get the anisotropic parameter as

$$\Delta = \frac{r^2(\rho+p_r)}{4(1+ar^2-br^2)}[p_r(1+ar^2)+b] - \frac{abnr^2(5+ar^2)}{(1+ar^2)^3} \tag{18}$$

which clearly shows that anisotropy vanishes at the centre in this model, i.e., $p_r = p_t$ at $r = 0$. The metric potentials are nonsingular at the centre. The energy density and the two pressures (p_r and p_t) are also well behaved in the stellar interior as will be shown later.

In our model, a, b, c have the dimension of *length*$^{-2}$. For simplicity in numerical calculations, we make the following transformations:

$$\tilde{a} = aR^2, \quad \tilde{b} = bR^2, \quad \tilde{c} = cR^2,$$

where, R is a parameter which has the dimension of a *length*. In terms of these dimensionless parameters our results may be summarized as follows:

$$e^{\mu} = \frac{1+\tilde{a}y}{1+(\tilde{a}-\tilde{b})y} \tag{19}$$

$$e^{\gamma} = B(1+\tilde{a}y)^n[1+(\tilde{a}-\tilde{b})y]^{\tilde{k}} \exp\left[-\frac{\tilde{a}\tilde{c}y}{2(\tilde{a}-\tilde{b})}\right] \tag{20}$$

$$\rho = \frac{\tilde{b}(3+\tilde{a}y)}{R^2(1+\tilde{a}y)^2} \tag{21}$$

$$p_r = n(\rho - \rho_s) \tag{22}$$

$$p_t = p_r + \Delta \tag{23}$$

where

$$\tilde{k} = \frac{5n\tilde{a}\tilde{b} - 2n\tilde{a}^2 - 3n\tilde{b}^2 + \tilde{a}\tilde{b} - \tilde{b}^2 - \tilde{b}\tilde{c}}{2(\tilde{a}-\tilde{b})^2} \tag{24}$$

and $y = r^2/R^2$. We may also write the anisotropic parameter in the form

$$\begin{aligned}\Delta = {} & \frac{r^2}{4R^4(1+\tilde{a}y-\tilde{b}y)(1+\tilde{a}y)^3} \times \\ & [n(n+1)\tilde{b}^2(3+\tilde{a}y)^2 - (2n+1)\tilde{b}\tilde{c}(3+\tilde{a}y)(1+\tilde{a}y)^2 \\ & -\tilde{c}^2(1+\tilde{a}y)^4 + \tilde{b}^2(n+1)(3+\tilde{a}y)(1+\tilde{a}y) \\ & -\tilde{b}\tilde{c}(1+\tilde{a}y)^3 - 4n\tilde{a}\tilde{b}(5+\tilde{a}y)(1+\tilde{a}y-\tilde{b}y)].\end{aligned} \tag{25}$$

The mass contained within a radius s is given by

$$M = \frac{\tilde{b}s^3/R^2}{2(1+\tilde{a}s^2/R^2)}. \tag{26}$$

3 PHYSICAL ANALYSIS

In this section, following Delgaty & Lake (1998), we impose some restrictions on the model to make it physically acceptable.

- The interior solution should be matched with the Schwarzschild exterior model at the boundary $r = s$, which gives

$$e^{\mu(r=s)} = \left(1 - \frac{2M}{s}\right)^{-1} \tag{27}$$

$$e^{\gamma(r=s)} = \left(1 - \frac{2M}{s}\right). \tag{28}$$

- The values of $\tilde{a}$, $\tilde{b}$ should be so chosen that the energy density ρ and the radial pressure p_r remain positive inside the star.
- The causality condition is satisfied if we restrict our model with $0 \leqslant n \leqslant 1$.
- The value of Δ should be so chosen that $\frac{dp_r}{dr} \leqslant 0$ in the interior of the star and the tangential pressure $p_t \geqslant 0$. This imposes a bound on the anisotropy given by

$$\Delta \leqslant \frac{r^2(\rho+p_r)}{4(1+(\tilde{a}-\tilde{b})r^2/R^2)}[p_r(1+\tilde{a}r^2/R^2)+\tilde{b}/R^2] \tag{29}$$

- The model has five independent parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, n and B. Note that R is not independent as it can be expressed in terms of $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. Two of the above parameters can be fixed by the matching conditions given by equations (27) and (28). The third parameter becomes fixed if we choose the central/surface density. Thus we will be left with two free parameters governing the geometry and EOS of the star.

By suitably choosing values of the unknown parameters, it is possible to show that our model can describe realistic compact stellar objects. For example, let us consider a particular case where the central density and the surface density are given by $\rho_c = 4.68 \times 10^{15}$ gm cm^{-3} and $\rho_s = 1.17 \times 10^{15}$ gm cm^{-3}, respectively. If we now choose, $\tilde{a} = 53.34$, $\tilde{b} = 54.34$ and $R = 43.245$ km, then the model yields a star of mass $M = 1.435\ M_{\odot}$ and radius $s = 7.07$ km. Note that these are the values which can be obtained if we use one of the EOS for strange matter formulated by Dey *et al* (1998). The behaviour of the energy density, two pressures and anisotropic parameter are shown in Figures 1-3, respectively. It is to be noted here that, unlike the slope in Dey *et al* (1998) (which was $n = 0.463$), the maximum slope for a physically reasonable model in this case is given by $n = 0.297$. This suggests that if we want to obtain the same mass and radius for the central and surface densities as in Dey *et al* (1998) (which arguably described the X-ray pulsar SAX J 1808.4-3658), and include anisotropy which vanishes at the boundary, then the EOS becomes much softer, i.e. the gradient $dp_r/d\rho$ decreases. On the other hand, keeping the same slope ($n = 0.463$) and surface density ($\rho_s = 1.17 \times 10^{15}$ gm cm^{-3}), will give us a star of mass $\sim 1.96\ M_{\odot}$. Choosing different sets of values will obviously give different results as shown in Table 1. Thus anisotropy provides a tool to consider a large class of degenerate states in this model.

4 DISCUSSION

We have obtained new exact solutions to the Einstein field equations capable of describing relativistic compact objects with anisotropic matter distribution and admitting a linear EOS. The solutions obtained are nonsingular and

Table 1. Central density and mass for different anisotropic stellar models. We have set, $s = 7.07$ km, $\rho_s = 1.17 \times 10^{15}$ gm cm^{-3} and $R = 43.245$ km.

$\tilde{b}$	$\tilde{a}$	n	ρ_c (gm cm^{-3})	Mass ($M_\odot$)
30	23.681	0.271	2.58	1.177
40	36.346	0.277	3.44	1.300
50	48.308	0.290	4.31	1.398
53.34	54.340	0.297	4.68	1.435
60	59.789	0.306	5.17	1.479
70	70.921	0.322	6.03	1.548
80	81.786	0.337	6.89	1.608
90	92.442	0.353	7.75	1.661
100	102.929	0.367	8.61	1.707
183	186.164	0.463	15.76	1.962

well behaved in the stellar interior. Unlike other models (Mak & Harko 2003; Dev & Gleiser 2004), the form of the anisotropic parameter is not chosen a priori in this model. We have shown that the consideration of anisotropy may provide a wide range of consequences in the geometry and composition of a star. For example, this may either yield a softer EOS (so that the gradient $dp_r/d\rho$ may be different) or a star with different mass and radius. Our analysis depends critically on the choice of the mass function given by equation (11) so that a linear equation of state is possible. Note that some other treatments, such as the result of Maharaj & Chaisi (2006), with a linear EOS has the unrealistic feature of vanishing energy density at the boundary. Our model does not suffer from this defect. It is also important to observe that in the presence of anisotropy the material composition, in objects such as SAX J1808.4-3658, need not necessarily be quark matter because the softness may vary. In future work it would be interesting to investigate what other forms of the mass function are consistent with a linear equation of state.

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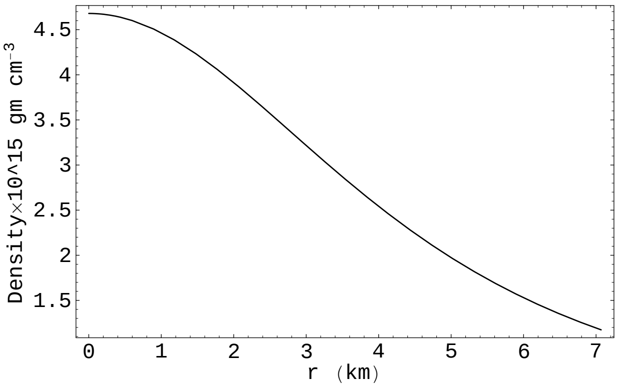

Figure 1. Energy density plotted against radial distance.

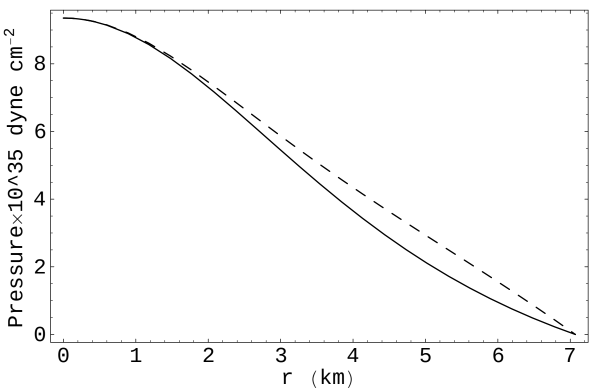

Figure 2. Radial pressure (solid line) and tangential pressure (dashed line) plotted against distance.

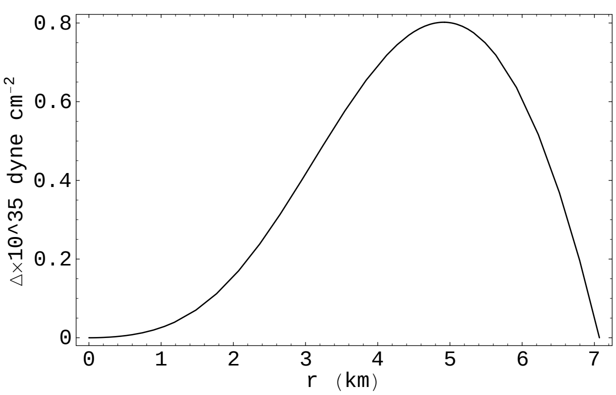

Figure 3. Anisotropic parameter plotted against radial distance.